On the minimal degree of finite permutation groups

Yi Lin Xie

Abstract. Let Ω be a non-empty finite set. For a permutation group $G \leq \mathrm{Sym}(\Omega)$ and an element $x \in G$, the degree $\deg(x)$ of x is the number of elements moved by x. The minimal degree of G, denoted by $\mu(G)$, is the minimum among the degrees of the elements in G.

Babai [1] proved that, for a primitive permutation group G, either $2\mu(G) \geq \sqrt{|\Omega|} - 1$ or G contains $\mathrm{Alt}(\Omega)$. Later, Liebeck and Saxl [8] improved this bound, showing that $\mu(G) \geq \frac{1}{2}|\Omega|$ unless G belongs to a specific list of exceptions. Guralnick and Magaard [7] subsequently provided a detailed classification of all primitive permutation groups G with $\mu(G) < \frac{1}{2}|\Omega|$. More recently, in a series of papers [2, 3, 4, 5], Burness studied the fixed-point ratio $1 - |\Omega|^{-1}\deg(x)$ of prime-order elements. Building on these results, Burness and Guralnick [6] further refined the bounds on the minimal degree by classifying all primitive permutation groups G with $\mu(G) \leq \frac{2}{3}|\Omega|$.

The inductive method of Guralnick–Magaard and its application to various parabolic subgroups remain excellent foundational material for studying classical groups and primitive actions, which is the focus of the talk. As an example, we will demonstrate how to apply this method to $L_n(2)$, the projective special linear group of degree n over the field with two elements.

About the speaker. Yi Lin Xie was a PhD student in finite and permutation group theory at SUSTech, under the supervision of Cai Heng Li. After defending her thesis on fixers of permutation groups, she is now exploring topics in combinatorial and geometric group theory with her new advisor, Alexander A. Ivanov, at HebNU. Her current research also focuses on matrix groups and combinatorial structures whose full automorphism group contains a Lie-type group of twisted rank one.

References

[1] L. Babai, *On the order of uniprimitive permutation groups*, Ann. of Math. (2) **113** (1981), no. 3, 553–568.
[2] T. C. Burness, *Fixed point ratios in actions of finite classical groups. I*, J. Algebra **309** (2007), no. 1, 69–79.
[3] T. C. Burness, *Fixed point ratios in actions in finite classical groups. II*, J. Algebra **309** (2007), no. 1, 80–138.
[4] T. C. Burness, *Fixed point ratios in actions of finite classical groups. III*, J. Algebra **314** (2007), no. 2, 693–748.
[5] T. C. Burness, *Fixed point ratios in actions of finite classical groups. III*, J. Algebra **314** (2007), no. 2, 693–748.
[6] T. C. Burness and R. M. Guralnick, *Fixed point ratios for finite primitive groups and applications*, Adv. Math. **411** (2022), Paper No. 108778, 90 pp.
[7] R. M. Guralnick and K. Magaard, *On the minimal degree of a primitive permutation group*, J. Algebra **207** (1998), no. 1, 127–145.
[8] M. W. Liebeck and J. Saxl, *Minimal degrees of primitive permutation groups, with an application to monodromy groups of covers of Riemann surfaces*, Proc. London Math. Soc. (3) **63** (1991), no. 2, 266–314.

School of Mathematical Sciences, Hebei Normal University | 050024 Shijiazhuang, China
E-mail: xyl_crazystu@hebtu.edu.cn ·